On the realization of transitive Novikov algebras

2001 J. Phys. A: Math. Gen. 34 3363

(http://iopscience.iop.org/0305-4470/34/16/303)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 3363–3372 www.iop.org/Journals/ja PII: S0305-4470(01)20888-9

On the realization of transitive Novikov algebras

Chengming Bai[1] **and Daoji Meng**[2]

[1] Theoretical Physics Division, Nankai Institute of Mathematics, Tianjin 300071, People's Republic of China

[2] Department of Mathematics, Nankai University, Tianjin 300071, People's Republic of China

Received 11 January 2001

Abstract

Novikov algebras were introduced in connection with the Poisson brackets of hydrodynamic-type and Hamiltonian operators in the formal variational calculus. It is well known that the radical of a finite-dimensional Novikov algebra is transitive. In this paper, we prove that a kind realization of Novikov algebras given by S Gel'fand is transitive and we give a deformation theory of Novikov algebras. In two and three dimensions, we find that all transitive Novikov algebras can be realized as the Novikov algebras given by S Gel'fand and their compatible infinitesimal deformations.

PACS numbers: 0210, 0220, 0240

1. Introduction

Hamiltonian operators are closely related to certain algebraic structures [1–8]. Gel'fand and Dikii introduced formal variational calculus and found certain interesting Poisson structures when they studied Hamiltonian systems related to certain nonlinear partial differential equations, such as KdV equations [1, 2]. In [3], further connections between Hamiltonian operators and certain algebraic structures were found. Dubrovin, Balanskii and Novikov studied similar Poisson structures from another point of view [4–6]. One of the algebraic structures appearing in [3, 6], which is called a 'Novikov algebra' by Osborn [9–14], was introduced in connection with the Poisson brackets of hydrodynamic type.

A Novikov algebra A is a vector space over a field $\boldsymbol{K}$ with a bilinear product $(x, y) \rightarrow xy$ satisfying

$$(x_1, x_2, x_3) = (x_2, x_1, x_3) \tag{1.1}$$

and

$$(x_1x_2)x_3 = (x_1x_3)x_2 \tag{1.2}$$

for $x_1, x_2, x_3 \in A$, where

$$(x_1, x_2, x_3) = (x_1x_2)x_3 - x_1(x_2x_3). \tag{1.3}$$

Novikov algebras are a special class of left-symmetric algebras which only satisfy equation (1.1). Left-symmetric algebras are non-associative algebras arising from the study of affine manifolds, affine structures and convex homogeneous cones [15–18].

The commutator of a Novikov algebra (or a left-symmetric algebra) A

$$[x, y] = xy - yx \tag{1.4}$$

defines a (sub-adjacent) Lie algebra $\mathcal{G} = \mathcal{G}(A)$. Let L_x, R_x denote left and right multiplication, respectively, i.e. $L_x(y) = xy, R_x(y) = yx, \forall x, y \in A$. Then for a Novikov algebra, the left multiplication operators form a Lie algebra and the right multiplication operators are commutative.

Zel'manov gave a fundamental structure theory of a finite-dimensional Novikov algebra over an algebraically closed field with characteristic 0 [19]: a Novikov algebra A is called right-nilpotent or transitive if every R_x is nilpotent. Then by equation (1.2), a finite-dimensional Novikov algebra A contains a (unique) largest transitive ideal $N(A)$ (called the radical of A) and the quotient algebra $A/N(A)$ is a direct sum of fields. The transitivity corresponds to the completeness of the affine manifolds in geometry [15, 16].

Therefore, it is necessary to understand the structures and properties of transitive Novikov algebras in detail. This is still an open question, which is obviously quite difficult. This can be seen from the complicated classification of Novikov algebras in low dimensions [20]. One of the reasons for this is due to there not being a 'suitable' representation theory for Novikov algebras because they are not associative in general (hence their representations should have bi-module structures). So it is important to find some realizations of transitive Novikov algebras at first, which will be useful to construct a general theory.

The first important kind of Novikov algebras was found by S Gel'fand [3]: let $(A, \cdot)$ be a commutative associative algebra, and D be its derivative. Then with the new product

$$a * b = a \cdot Db \tag{1.5}$$

$(A, *)$ becomes a Novikov algebra. Later, Filipov [21] proved that for any $\xi \in \boldsymbol{K}$, the product

$$a *_\xi b = a \cdot Db + \xi a \cdot b \tag{1.6}$$

makes $(A, *_\xi)$ into a Novikov algebra, too. Xu [13] extended ξ to a fixed element $a \in A$, i.e. with the product

$$a *_x b = a \cdot Db + x \cdot a \cdot b \tag{1.7}$$

$(A, *_x)$ is still a Novikov algebra. Furthermore, using the product defined by (1.7), Xu found several classes of infinite-dimensional simple Novikov algebras [13].

In the paper, we discuss the finite-dimensional Novikov algebras defined by the above equations. The paper is organized as follows. In section 2, we find that the Novikov algebra defined by equation (1.5) is transitive. In section 3, we give a deformation theory for Novikov algebras and the Novikov algebra defined by equation (1.6) or equation (1.7) can be regarded as a deformation of the algebra defined by equation (1.5). In section 4, we can find that all the transitive Novikov algebras in three dimensions can be realized as the algebras defined by equation (1.5) and their compatible infinitesimal deformations. In section 5, we give some conjectures based on the discussion in the previous sections.

2. The Novikov algebras defined by equation (1.5)

As in the introduction, let $(A, \cdot)$ be a finite-dimensional commutative associative algebra. Let D be its derivative, i.e.

$$D(a \cdot b) = Da \cdot b + a \cdot Db \qquad \forall\, a, b \in A. \tag{2.1}$$

$(A, *)$ is a Novikov algebra with the product

$$a * b = a \cdot Db. \tag{2.2}$$

Let R be the radical of $(A, \cdot)$, that is, R is the maximal nilpotent ideal of $(A, \cdot)$. By Wedderburn's principal theorem [22], there exists a subalgebra S such that

$$A/R \cong S \qquad \text{and} \qquad A = R + S \tag{2.3}$$

and S is isomorphic to the direct sum of fields. Hence, we can choose a basis $\{e_1, e_2, \ldots, e_n\}$ such that $\{e_1, \ldots, e_k\} \subset R$, $\{e_{k+1}, \ldots, e_n\} \subset S$, and

$$\begin{array}{llll} e_i e_j \in R & e_i e_\alpha \in R & e_\alpha e_\beta = \delta_{\alpha\beta} e_\alpha + a_{\alpha\beta} & a_{\alpha\beta} \in R \\ 1 \leqslant i & j \leqslant k & k+1 \leqslant \alpha & \beta \leqslant n. \end{array} \tag{2.4}$$

Claim. *For any derivation D, we have $D(A) \subset R$.*

It is well known (see [23], chapter I, exercise 22) that for an associative algebra the derivation maps the radical into the radical, that is, $D(R) \subset R$. For $e_\alpha (k+1 \leqslant \alpha \leqslant n)$, set

$$De_\alpha = \sum_{i=1}^{k} d_i e_i + \sum_{\gamma=k+1}^{n} d_\gamma e_\gamma. \tag{2.5}$$

Then by equations (2.4) and (2.1), we have

$$D(e_\alpha \cdot e_\alpha) = 2e_\alpha \cdot D(e_\alpha) = D(e_\alpha + a_{\alpha\alpha}). \tag{2.6}$$

Comparing the coefficients of e_γ, $k+1 \leqslant \gamma \leqslant n$, we have

$$d_\gamma = 0 \qquad k+1 \leqslant \gamma \leqslant n. \tag{2.7}$$

So this implies that $D(e_\alpha) \in R$. Therefore, $D(A) \subset R$.

For any $a \in (A, \cdot)$, $Da \in R$ is nilpotent since $R \subset (A, \cdot)$ is nilpotent. Hence there exists $n \in \mathbf{Z}$ such that $(Da)^n = 0$. Thus in $(A, *)$, we have

$$(R_a)^n(b) = ((\cdots(b * a) * a) \cdots * a) * a = b \cdot (Da)^n = 0 \qquad \forall b \in A. \tag{2.8}$$

This means that R_a is a nilpotent transformation of $(A, *)$. Hence $(A, *)$ is transitive.

Since the sub-adjacent Lie algebra $\mathcal{G}(A)$ of a transitive left-symmetric algebra A is solvable [15], we have

Corollary. *Let $(A, \cdot)$ be a commutative associative algebra, D be its derivative. Then the algebra defined by*

$$[a, b] = a \cdot Db - b \cdot Da \tag{2.9}$$

is a solvable Lie algebra.

Example 2.1. There are three non-isomorphic transitive Novikov algebras in two dimensions: (T1), (T2) and (T3) [20]. All of them can be realized as the algebras defined by equation (1.5), which can be seen from table 1: recall that the (form) characteristic matrix of a Novikov algebra is defined as

$$\mathcal{A} = \begin{pmatrix} \sum_{k=1}^n a_{11}^k e_k & \cdots & \sum_{k=1}^n a_{1n}^k e_k \\ \cdots & \cdots & \cdots \\ \sum_{k=1}^n a_{n1}^k e_k & \cdots & \sum_{k=1}^n a_{nn}^k e_k \end{pmatrix} \tag{2.10}$$

Table 1.

Characteristic matrix of $(A, \cdot)$	Derivation algebras of $(A, \cdot)$	Characteristic matrix of $(A, *)$ under $D \neq 0$ (isomorphic classes)
(T1) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$gl(2) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$	(T1) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
(T2) $\begin{pmatrix} e_2 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 \\ a_{21} & 2a_{11} \end{pmatrix}$	(T2) $\begin{pmatrix} e_2 & 0 \\ 0 & 0 \end{pmatrix}$ $(a_{21} = 0)$
		(T3) $\begin{pmatrix} 0 & 0 \\ -e_1 & 0 \end{pmatrix}$ $(a_{21} \neq 0)$
(N1) $\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}$	0	(T1)
(N2) $\begin{pmatrix} e_1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & a_{22} \end{pmatrix}$	(T1)
(N3) $\begin{pmatrix} e_1 & e_2 \\ e_2 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & a_{22} \end{pmatrix}$	(T2)

where $\{e_i\}$ is a basis of A and $e_i e_j = \sum_{k=1}^{n} a_{ij}^k e_k$. Moreover, under the same basis, any derivation D of A can be determined by a matrix, that is,

$$D = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \qquad D(e_i) = \sum_{j=1}^{n} a_{ij} e_j. \tag{2.11}$$

There exist transitive Novikov algebras which cannot be realized as the algebras defined by equation (1.5). We will give such examples in section 4.

3. The deformations of Novikov algebras

A method to obtain new Novikov algebras is the study of deformations of Novikov algebras. A general theory for deformations is given in [24].

Let $(A, *)$ be a Novikov algebra, and $g_p : A \times A \to A$ be a bilinear product defined by

$$g_q(a, b) = a * b + qG_1(a, b) + q^2 G_2(a, b) + q^3 G_3(a, b) + \cdots \tag{3.1}$$

where G_i are bilinear products with $G_0(a, b) = a * b$. (A_q, g_q) is a family of Novikov algebras if and only if

$$g_q(a, g_q(b, c)) - g_q(g_q(a, b), c)) = g_q(b, g_q(a, c)) - g_q(g_q(b, a), c)) \tag{3.2}$$

$$g_q(g_q(a, b), c)) = g_q(g_q(a, c), b)) \tag{3.3}$$

for all $a, b, c \in A$. The two equations are equivalent to the following two equations:

$$\sum_{m+n=k} [G_m(a, G_n(b, c)) - G_m(G_n(a, b), c) - G_m(b, G_n(a, c)) + G_m(G_n(b, a), c)] = 0 \tag{3.4}$$

$$\sum_{m+n=k} [G_m(G_n(a, b), c) - G_m(G_n(a, c), b)] = 0 \tag{3.5}$$

for all non-negative integers k, and $m, n \geqslant 0$. For $k = 0$ this means $A = A_0$ is a Novikov algebra. For $k = 1$ we obtain two equations for G_1:

$$G_1(a, b * c) - G_1(a * b, c) + G_1(b * a, c) - G_1(b, a * c) + a * G_1(b, c) - G_1(a, b) * c$$
$$+G_1(b, a) * c - b * G_1(a, c) = 0 \tag{3.6}$$

$$G_1(a, b) * c - G_1(a, c) * b + G_1(a * b, c) - G_1(a * c, b) = 0. \tag{3.7}$$

We call G_1 an infinitesimal deformation. Since the commutator of a Novikov algebra is a Lie algebra, we should consider some kind of 'compatible' properties for the deformations. Obviously, the algebra (A_q, g_q) defined by an infinitesimal deformation G_1:

$$g_q(a, b) = a * b + qG_1(a, b) \tag{3.8}$$

has the same sub-adjacent Lie algebra structure $\mathcal{G}(A)$ with $(A, *)$ if and only if G_1 is a commutative algebra, i.e.

$$G_1(a, b) = G_1(b, a). \tag{3.9}$$

In this case, we call G_1 a compatible (infinitesimal) deformation.

Example 3.1. Both the families of Novikov algebras $(A, *_\xi)$ defined by equation (1.6) and $(A, *_{tx})$ defined by equation (1.7) are compatible infinitesimal deformations of the Novikov algebra defined by equation (1.5) with a fixed derivative D. For equation (1.6), we let $G_1(a, b) = a \cdot b$, then equation (3.6) holds since

$$a \cdot (b \cdot Dc) - (a \cdot Db) \cdot c + b \cdot Da \cdot c - b \cdot (a \cdot Dc) = b \cdot c \cdot Da - a \cdot Db \cdot c$$
$$a \cdot D(b \cdot c) - a \cdot b \cdot Dc + b \cdot a \cdot Dc - b \cdot D(a \cdot c) = a \cdot Db \cdot c - b \cdot c \cdot Da$$

and equation (3.7) holds since

$$a \cdot b \cdot Dc - a \cdot c \cdot Db + a \cdot Db \cdot c - a \cdot Dc \cdot b = 0.$$

For equation (1.7), we let $G_1(a, b) = x \cdot a \cdot b$. Similarly, it is easy to verify that equations (3.6) and (3.7) hold. And the algebras defined by equations (1.5)–(1.7) have the same sub-adjacent Lie algebras from the commutativity of $(A, \cdot)$.

A natural question to ask is when two deformations are equivalent, which means that the resulting Novikov algebras are isomorphic? This question can be answered by a cohomology theory of Novikov algebras, which will be discussed in detail elsewhere [25]. In this paper, we only briefly discuss the 2-cohomology group. In fact, the infinitesimal deformation G_1 belongs to the so-called 2-cocycles: let $(A, *)$ be a Novikov algebra, then the space of 2-cocycles and 2-coboundaries is given by

$$Z^2(A, A) = \{f : A \times A \to A | f \text{ is bilinear; equations (3.6) and (3.7) hold}\} \tag{3.10}$$

$$B^2(A, A) = \{f : A \times A \to A | f \text{ is bilinear; } f(a, b) = a * g(b) + g(a) * b - g(a * b),$$
$$\text{for some } g : A \to A \text{ and } g \text{ is linear}\}. \tag{3.11}$$

It is easy to show that $B^2(A, A) \subset Z^2(A, A)$ and we can give the 2-cohomology group $H^2(A, A)$ as

$$H^2(A, A) = Z^2(A, A)/B^2(A, A). \tag{3.12}$$

Two deformations of $(A, *)$ are equivalent if the 2-cocycles G_1 and G_1' are cohomological, that is $G_1 - G_1' \in B^2(A, A)$.

Example 3.2. For the Novikov algebras defined by equation (1.5), there exist compatible deformations which are not equivalent to those given by equation (1.6) and (1.7). Such an example can be seen in the next section.

At the end of this section, we would like to make the following remarks.

(a) It is obvious that any Novikov algebra can be regarded as a deformation of the trivial Novikov algebra (where all products are zero). In general, we assume that $(A, *)$ is not trivial.
(b) Usually, the deformation of a transitive Novikov algebra is not transitive. However, for a nilpotent commutative associative algebra A, the algebras defined by equations (1.5)–(1.7) are transitive.
(c) In general, we can obtain a family of Novikov algebras (A_q, g_q) through a deformation of a Novikov algebra A. Sometimes, this family is mutually isomorphic for $q \neq 0$. In such a case, we call it a special deformation.

4. The transitive Novikov algebras in three dimensions

In this section, we can see that all three-dimensional transitive Novikov algebras can be realized as the algebras defined by equation (1.5) and their compatible infinitesimal deformations. Firstly, we give all the transitive Novikov algebras defined by equation (1.5).

Remark. For the case $(A, \cdot)$ is (C11), the characteristic matrix of $(A, *)$ is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_{11}e_1 + a_{12}e_2 & a_{21}e_1 + a_{22}e_2 & 0 \end{pmatrix}$$

which the classification in table 2 is obtained in [20].

According to the classification of transitive Novikov algebras in three dimensions [20], there are the following algebras which cannot be the forms $(A, *)$ defined by equation (1.5).

(1) Type (A4):

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_1 \\ 0 & e_1 & e_2 \end{pmatrix}.$$

This algebra is isomorphic to the algebras defined by equation (1.6), where $(A, \cdot)$ is (A4),

$$D = \begin{pmatrix} 3a_{33} & 0 & 0 \\ 2a_{32} & 2a_{33} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

with $a_{33} = 0, a_{32} \neq 0$. In fact, this is a special deformation.

(2) Types (A7) with $l \neq 2$:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_1 \\ 0 & le_1 & e_2 \end{pmatrix} \qquad (l \neq 2, 1).$$

Table 2.

Characteristic matrix of $(A,\cdot)$	Derivation algebras of $(A,\cdot)$	Characteristic matrix of $(A,*)$ under $D\neq 0$ (isomorphic classes)
(A1) $\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$	$gl(3)=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}$	(A1) $\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$
(A2) $\begin{pmatrix}0&0&0\\0&0&0\\0&0&e_1\end{pmatrix}$	$\begin{pmatrix}2a_{33}&0&0\\a_{21}&a_{22}&0\\a_{31}&a_{32}&a_{33}\end{pmatrix}$	(A1) $(a_{33}=0)$
		(A2) $\begin{pmatrix}0&0&0\\0&0&0\\0&0&e_1\end{pmatrix}$ $(a_{33}\neq 0)$
(A3) $\begin{pmatrix}0&0&0\\0&e_1&0\\0&0&e_1\end{pmatrix}$	$\begin{pmatrix}2a_{22}&0&0\\a_{21}&a_{22}&a_{23}\\a_{31}&-a_{23}&a_{22}\end{pmatrix}$	(A1) $(a_{22}=a_{23}=0)$
		(A3) $\begin{pmatrix}0&0&0\\0&e_1&0\\0&0&e_1\end{pmatrix}$ $(a_{23}=0, a_{22}\neq 0)$
		(A5) $\begin{pmatrix}0&0&0\\0&0&e_1\\0&-e_1&0\end{pmatrix}$ $(a_{22}=0, a_{23}\neq 0)$
		(A6) $(l\neq 0)$ $\begin{pmatrix}0&0&0\\0&e_1&e_1\\0&-e_1&le_1\end{pmatrix}$ $(a_{22}\neq 0, a_{23}\neq 0)$
(A4) $\begin{pmatrix}0&0&0\\0&0&e_1\\0&e_1&e_2\end{pmatrix}$	$\begin{pmatrix}3a_{33}&0&0\\2a_{32}&2a_{33}&0\\a_{31}&a_{32}&a_{33}\end{pmatrix}$	(A1) $(a_{33}=a_{32}=0)$
		(A2) $(a_{33}=0, a_{32}\neq 0)$
		(A7) $(l=2)$ $\begin{pmatrix}0&0&0\\0&0&e_1\\0&2e_1&e_2\end{pmatrix}$ $(a_{33}\neq 0)$
$\begin{pmatrix}e_1&0&0\\0&e_2&0\\0&0&e_3\end{pmatrix}$	$\{0\}$	(A1)
(B1) $\begin{pmatrix}0&0&0\\0&e_2&0\\0&0&e_3\end{pmatrix}$	$\begin{pmatrix}a_{11}&0&0\\0&0&0\\0&0&0\end{pmatrix}$	(A1)
(B2) $\begin{pmatrix}0&0&e_1\\0&e_2&0\\e_1&0&e_3\end{pmatrix}$	$\begin{pmatrix}a_{11}&0&0\\0&0&0\\0&0&0\end{pmatrix}$	(A9) $\begin{pmatrix}0&0&0\\0&0&0\\0&e_2&0\end{pmatrix}$
(C1) $\begin{pmatrix}0&0&0\\0&0&0\\0&0&e_3\end{pmatrix}$	$\begin{pmatrix}a_{11}&a_{12}&0\\a_{21}&a_{22}&0\\0&0&0\end{pmatrix}$	(A1)
(C2) $\begin{pmatrix}0&0&e_1\\0&0&0\\e_1&0&e_3\end{pmatrix}$	$\begin{pmatrix}a_{11}&0&0\\0&a_{22}&0\\0&0&0\end{pmatrix}$	(A9)

These algebras belong to the family of algebras defined by equation (1.6), where $(A,\cdot)$ is (A4),

$$D=\begin{pmatrix}3a_{33}&0&0\\2a_{32}&2a_{33}&0\\a_{31}&a_{32}&a_{33}\end{pmatrix}$$

with $a_{33}\neq 0$.

Table 2. Continued.

Characteristic matrix of $(A, \cdot)$	Derivation algebras of $(A, \cdot)$	Characteristic matrix of $(A, *)$ under $D \neq 0$ (isomorphic classes)
(C11) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & e_2 & e_3 \end{pmatrix}$	$\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$	(A9)
		(A6) $(l=-1)$ $\begin{pmatrix} 0 & 0 & 0 \\ 0 & e_1 & e_1 \\ 0 & -e_1 & -e_1 \end{pmatrix}$
		(A11) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_1 & le_2 & 0 \end{pmatrix}$ $\lvert l\rvert \leqslant 1, l \neq 0$
		(A12) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_1 & e_1+e_2 & 0 \end{pmatrix}$
(D1) $\begin{pmatrix} e_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$	$\begin{pmatrix} a_{11} & a_{12} & 0 \\ 0 & 2a_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}$	(A2)
(D2) $\begin{pmatrix} e_2 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & e_2 & e_3 \end{pmatrix}$	$\begin{pmatrix} a_{11} & a_{12} & 0 \\ 0 & 2a_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}$	(A6) $(l=-1)$ $\begin{pmatrix} 0 & 0 & 0 \\ 0 & e_1 & e_1 \\ 0 & -e_1 & -e_1 \end{pmatrix}$ $(a_{11}=0)$
		(A13) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & e_1 & 0 \\ e_1 & \frac{1}{2}e_2 & 0 \end{pmatrix}$ $(a_{11} \neq 0)$

(3) Type (A6) with $l = 0$:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & e_1 & e_1 \\ 0 & -e_1 & 0 \end{pmatrix}.$$

We can show that this algebra cannot be obtained from equation (1.6) or equation (1.7). It is isomorphic to a special (compatible) infinitesimal deformation of (A5) with $G_1 =$ (A2).

(4) Type (A8):

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e_1 & e_2 \end{pmatrix}.$$

This algebra cannot be obtained from equation (1.6) or equation (1.7). It is isomorphic to a special (compatible) infinitesimal deformation of

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e_1 & 0 \end{pmatrix}.$$

(Type (A6) with $l = -1$) with

$$G_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e_2 \end{pmatrix}$$

which is isomorphic to (A2).

(5) Type (A10):

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e_2 & e_1 \end{pmatrix}.$$

This algebra cannot be obtained from equation (1.6) or equation (1.7). It is isomorphic to a special (compatible) infinitesimal deformation of (A9) with $G_1 = (\mathrm{A2})$.

5. Summary and conjectures

We have seen that in two and three dimensions, all transitive Novikov algebras can be realized as the algebras defined by equation (1.5) and their compatible infinitesimal deformations. We also find that some deformations are special. A natural question is whether these results can be extended to higher dimensions?

Conjecture 1. *All transitive Novikov algebras can be realized as the algebras defined by equation (1.5) and their compatible infinitesimal deformations.*

A direct corollary from the conjecture is

Conjecture 2. *The sub-adjacent Lie algebra of a Novikov algebra A is solvable and can be defined as*

$$[a, b] = a \cdot Db - b \cdot Da$$

where $(A, \cdot)$ *is a commutative associative algebra and* D *is its derivative.*

It is also interesting to know the possible application of these realizations in physics.

Acknowledgments

This work was supported in part by the National Natural Science Foundation of China, the Project for Young Mainstay Teachers and the Scientific Research Foundation for the Returned Overseas Chinese Scholars, State Education Ministry of China. We thank Professor X Xu for communicating with us his research in this field. The second author also thanks Professor Xu for the hospitality extended to him during his stay at Hong Kong University of Science and Technology and for valuable discussions.

References

[1] Gel'fand I M and Diki L A 1975 *Russ. Math. Surv.* **30** 77–113
[2] Gel'fand I M and Diki L A 1976 *Funct. Anal. Appl.* **10** 16–22
[3] Gel'fand I M and Dorfman I Ya 1979 *Funct. Anal. Appl.* **13** 248–62
[4] Dubrovin B A and Novikov S P 1983 *Sov. Math. Dokl.* **27** 665–9
[5] Dubrovin B A and Novikov S P 1984 *Sov. Math. Dokl.* **30** 651–4
[6] Balinskii A A and Novikov S P 1985 *Sov. Math. Dokl.* **32** 228–31
[7] Xu X 1995 *Lett. Math. Phys.* **33** 1–6
[8] Xu X 1995 *J. Phys. A: Math. Gen.* **28** 1681–98
[9] Osborn J M 1992 *Nova J. Algebra Geom.* **1** 1–14
[10] Osborn J M 1992 *Commun. Algebra* **20** 2729–53
[11] Osborn J M 1994 *J. Algebra* **167** 146–67
[12] Xu X 1996 *J. Algebra* **185** 905–34

[13] Xu X 1997 *J. Algebra* **190** 253–79
[14] Xu X 2000 *Preprint* math.QA/0008223
[15] Kim H 1986 *J. Diff. Geom.* **24** 373–94
[16] Vinberg E B 1963 *Transl. Moscow Math. Soc.* **12** 340–403
[17] Bai C M and Meng D J 2000 *Commun. Algebra* **28** 2717–34
[18] Burde D 1998 *Manuscipta Math.* **95** 397–411
[19] Zel'manov E I 1987 *Sov. Math. Dokl.* **35** 216–8
[20] Bai C M and Meng D J 2001 *J. Phys. A: Math. Gen.* **34** 1581–94
[21] Filipov V T 1989 *Mat. Zametki* **45** 101–5
[22] Albert A A 1961 *Structures of Algebra (AMS Coll. Pub XXIV)* (Providence, RI: American Mathematical Society)
[23] Jacobson N 1962 *Lie Algebras* (New York: Interscience)
[24] Gerstenhaber M 1964 *Ann. Math.* **79** 59–104
[25] Bai C M and Meng D J The cohomology groups of Novikov algebras *Preprint* Nankai